DUHAMEL INTEGRAL AND THE OPERATIONAL-STRUCTURAL METHOD OF SOLUTION IN NONSTATIONARY HEAT-CONDUCTION PROBLEMS FOR REGIONS OF COMPLICATED SHAPE

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We propose an analytic method for solving nonstationary heat-conduction problems for regions of complicated shape with nonstationary boundary conditions and energy sources.

In the solution of nonstationary heat-conduction problems for nonclassical regions by the operational-structural method [1], we make simultaneous use of the Laplace transform and the structural method. The latter enables us — when the Ritz method or the Bubnov–Galerkin method is used for solving the problem in the region of the mappings — to overcome the difficulties of constructing a system of coordinate functions that arise as a result of the complicated geometric shape of a constructional element and the nature of the boundary conditions on its surface. However, when the temperature of the external medium, the heat flux, and the intensity of the internal sources of energy vary with time in a complex manner, we encounter serious difficulties in connection with the use of the Laplace transform [2]. In the present article the solution of the heat-conduction problem with complicated nonstationary boundary conditions and energy sources is derived from the use of the operational-structural method in the form of Duhamel integrals [3]. As a result, when we have obtained the solution of the original problem in analytic form, we can retain in it the parameters characterizing the nonstationarity of the temperature of the external medium and the heat flux and the intensity of the internal energy sources.

We consider the problem of the distribution of the temperature field $u(M, t)$ in the region Ω when time varies in the interval $0 < t < \infty$:

$$\rho \frac{\partial u(M, t)}{\partial t} = Au(M, t) + \sum_{m_1=1}^{r} F_{m_1}(M) Q_{m_1}(t); \; u(M, t)\Big|_{t=0} = 0, \tag{1}$$

$$L_j u(M, t)\Big|_{\Gamma_j} = \sum_{m_2=1}^{l_j} f_{jm_2}(M) q_{jm_2}(t), \quad j = 1, \ldots, s, \tag{2}$$

where $Au(M, t) = \lambda[\Delta u(M, t) - \xi(M)u(M, t)]$; L_j is a first-order linear differential operator not dependent on t; $\bigcup_{j=1}^{s} \Gamma_j = \partial\Omega$. To the equation and the boundary conditions (1), (2) we apply the Laplace integral transform with respect to the variable t:

$$\bar{u}(M, p) \doteqdot u(M, t); \quad \bar{Q}_{m_1}(p) = Q_{m_1}(t); \quad \bar{q}_{jm_2}(p) = q_{jm_2}(t).$$

Then in the region of mappings, we obtain

$$A\bar{u}(M, p) - p\rho\bar{u}(M, p) + \sum_{m_1=1}^{r} F_{m_1}(M) \bar{Q}_{m_1}(p) = 0 \tag{3}$$

$$L_j \bar{u}(M, p)\Big|_{\Gamma_j} = \sum_{m_2=1}^{l_j} f_{jm_2}(M) \bar{q}_{jm_2}(p), \; j = 1, \ldots, s. \tag{4}$$

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The solution of problem (3), (4), in accordance with [1], will be sought in the form

$$\bar{u}^{(n)}(M,\ p)=\sum_{j=1}^{s}\sum_{m_1=1}^{l_j}\Phi_{jm_2}(M)\,q_{jm_2}(p)+\sum_{i=1}^{n}\bar{C}_i(p)\,\chi_i(M), \tag{5}$$

where the $\chi_i(M)$ are coordinate functions satisfying homogeneous boundary conditions of the problem (3), (4), and $\Phi_{jm_2}(M)$ satisfies the conditions

$$L_j\Phi_{jm_2}(M)\Big|_{\Gamma_k}=\begin{cases} f_{jm_2}(M), & k=j,\\ 0 & , k\neq j.\end{cases}$$

The Bubnov–Galerkin system for determining the mapping coefficients $\bar{C}_i(p)$ has the form

$$\sum_{i=1}^{n}(A_{ik}+pB_{ik})\,\bar{C}_i(p)=E_k(p), \tag{6}$$

where

$$A_{ik}=\lambda\int_{\Omega}(\Delta\chi_i-\xi\chi_i)\,\chi_k d\Omega;\quad B_{ik}=-\int_{\Omega}\rho\chi_i\chi_k d\Omega;$$

$$E_k(p)=\sum_{m_1=1}^{r}\bar{Q}_{m_1}(p)\,\alpha_{km_1}+\sum_{j=1}^{s}\sum_{m_2=1}^{l_j}q_{jm_2}(p)\,\beta_{kjm_2};$$

$$\alpha_{km_1}=-\int_{\Omega}F_{m_1}\chi_k d\Omega;\quad \beta_{kjm_2}=\int_{\Omega}[-\mathrm{A}\Phi_{jm_2}+p\rho\Phi_{jm_2}]\,\chi_k d\Omega.$$

For the solution of system (6) we obtain

$$\bar{C}_i(p)=\Big[\sum_{k=1}^{n}E_k(p)\,\Delta_{ik}(p)\Big]\,[\Delta(p)]^{-1}, \tag{7}$$

where $\Delta(p)$ is the determinant of the matrix of the system (6) and the $\Delta_{ik}(p)$ are the corresponding cofactors. We represent (7) in the form

$$\bar{C}_i(p)=\Big[\sum_{k=1}^{n}\Big(\sum_{m_1=1}^{r}Q_{m_1}(p)\,\alpha_{km_1}+\sum_{j=1}^{s}\sum_{m_2=1}^{l_j}q_{jm_2}(p)\,\beta_{kjm_2}\Big)\Delta_{ik}(p)\Big][\Delta(p)]^{-1}=$$

$$=\sum_{m_1=1}^{r}p\bar{Q}_{m_1}(p)\sum_{k=1}^{n}\alpha_{km_1}\frac{\Delta_{ik}(p)}{p\Delta(p)}+\sum_{j=1}^{s}\sum_{m_2=1}^{l_j}pq_{jm_2}(p)\sum_{k=1}^{n}\beta_{kjm_2}\frac{\Delta_{ik}(p)}{p\Delta(p)}=$$

$$=\sum_{m_1=1}^{r}p\bar{Q}_{m_1}(p)\,\bar{C}^{\mathrm{I}}_{im_1}(p)+\sum_{j=1}^{s}\sum_{m_2=1}^{l_j}pq_{jm_2}(p)\,\bar{C}^{\mathrm{II}}_{ijm_2}(p).$$

Making use of the theorem on the differentiation and convolution of the originals [4], we obtain

$$C_i(t)=\sum_{m_1=1}^{r}\int_0^t Q_{m_1}(\tau)\frac{d}{dt}C^{\mathrm{I}}_{im_1}(t-\tau)\,d\tau+\sum_{j=1}^{s}\sum_{m_2=1}^{l_j}\int_0^t q_{jm_2}(\tau)\frac{d}{dt}C^{\mathrm{II}}_{ijm_2}(t-\tau)\,d\tau,$$

where

$$\bar{C}_i(p)\doteqdot C_i(t);\quad \bar{C}^{\mathrm{I}}_{im_1}(p)\doteqdot C^{\mathrm{I}}_{im_1}(t);\quad \bar{C}^{\mathrm{II}}_{ijm_2}(p)\doteqdot C^{\mathrm{II}}_{ijm_2}(t)$$

The solution of the initial problem (1), (2) is found from (5):

$$u^{(n)}(M,\ t)=\sum_{m_1=1}^{r}\int_0^t Q_{m_1}(\tau)\frac{d}{dt}\sum_{i=1}^{n}C^{\mathrm{I}}_{im_1}(t-\tau)\chi_i(M)\,d\tau+$$

$$+\sum_{j=1}^{s}\sum_{m_2=1}^{l_j}\left[\int_0^t q_{jm_2}(\tau)\frac{d}{dt}\sum_{i=1}^{n}C^{\mathrm{II}}_{ijm_2}(t-\tau)\chi_i(M)\,d\tau+\Phi_{jm_2}(M)\,q_{jm_2}(t)\right]=$$

$$=\sum_{m_1=1}^{r}\int_0^t Q_{m_1}(\tau)\frac{\partial}{\partial t}W^{(n)}_{m_1}(M,\ t-\tau)\,d\tau+ \tag{8}$$

$$+\sum_{j=1}^{s}\sum_{m_2=1}^{l_j}\left[\int_0^t q_{jm_2}(\tau)\frac{\partial}{\partial t}V^{(n)}_{jm_2}(M,\ t-\tau)\,d\tau+q_{jm_2}(0)\,f_{jm_2}(M)\right],$$

where

$$W^{(n)}_{m_1}(M,\ t)=\sum_{i=1}^{n}C^{\mathrm{I}}_{im_1}(t)\,\chi_i(M) \tag{9}$$

is the solution of the problem

$$\frac{\partial u(M,\ t)}{\partial t}=\mathrm{A}u(M,\ t)+F_{m_1}(M);\quad u(M,\ t)\Big|_{t=0}=0; \tag{10}$$
$$L_iu(M,\ t)|_{\Gamma_i}=0,\quad i=1,\ \ldots,\ s,$$

and

$$V^{(n)}_{jm_2}(M,\ t)=\sum_{i=1}^{n}C^{\mathrm{II}}_{ijm_2}(t)\,\chi_i(M)+\Phi_{jm_2}(M) \tag{11}$$

is the solution of the problem

$$\frac{\partial u(M,\ t)}{\partial t}=\mathrm{A}u(M,\ t);\quad u(M,\ t)\Big|_{t=0}=0; \tag{12}$$
$$L_iu(M,\ t)\Big|_{\Gamma_i}=\begin{cases}f_{jm_2}(M), & i=j,\\ 0, & i\neq j,\end{cases}\quad i=1,\ \ldots,\ s.$$

Thus, solutions (9), (11) are constructed by means of the operational-structural method with a unified system of coordinate functions $\chi_i(M)$, and the solution of the initial problem (1), (2) is presented in the form of Duhamel integrals through solutions (9), (11) of problems (10), (12). The representation of the solution of the initial problem in form (8) makes it unnecessary to find the mappings for the functions $Q_{m_1}(t)$, $q_{jm_2}(t)$, makes the inverse transform a stereotypical procedure, and makes it possible to carry out an analysis of the solution of the initial problem for different $Q_{m_1}(t)$, $q_{jm_2}(t)$.

On the basis of the above-described method for solving nonstationary heat-conduction problems for nonclassical regions, we developed algorithms which were set up in the form of programs on the BESM-6 computer. A test of the algorhtms, which was carried out for problems with different nonstationary boundary conditions and nonstationary energy sources, showed that the error in the calculation of the temperature fields is determined in practice by the complicated nature of the geometric information and is almost independent of the form of the nonstationary components of the boundary conditions and the functions for the energy sources $Q_{m_1}(t)$, $q_{jm_2}(t)$.

As an example of the use of this method, let us consider a case in which the determination of the temperature field u(x, y, t) in a plate containing a system of energy sources (Fig. 1) reduces to the solution of the following nonstationary heat-conduction problem:

$$\frac{\partial u(x,\ y,\ \mathrm{Fo})}{\partial\,\mathrm{Fo}} = \Delta u(x,\ y,\ \mathrm{Fo}) - b^2 u(x,\ y,\ \mathrm{Fo}) + \sum_{m=1}^{3} f_m(x,\ y)\,\varphi_m(\mathrm{Fo}), \tag{13}$$

$$u(x,\ y,\ \mathrm{Fo})\Big|_{\mathrm{Fo}=0} = 0,\quad f_m(x,\ y) = \begin{cases} \dfrac{P}{0{,}01\lambda d}, & (x,\ y) \in D_m, \\ 0, & (x,\ y) \overline{\in} D_m, \end{cases}$$

$$u(x,\ y,\ \mathrm{Fo})\Big|_{\Gamma_1} = 0,\qquad \frac{\partial u(x,\ y,\ \mathrm{Fo})}{\partial \nu}\Big|_{\Gamma_2} = 0. \tag{14}$$

The solution of problem (13), (14), in accordance with (8), can be represented in the form

$$u^{(n)}(x,\ y,\ \mathrm{Fo}) = \sum_{m=1}^{3} \int_0^{\mathrm{Fo}} \varphi_m(\tau)\,\frac{\partial}{\partial\,\mathrm{Fo}}\,W_m^{(n)}(x,\ y,\ \mathrm{Fo}-\tau)\,d\tau, \tag{15}$$

where

$$W_m^{(n)}(x,\ y,\ \mathrm{Fo}) = \sum_{i,j} C_{ijm}(\mathrm{Fo})\,\chi_{ij}(x,\ y)$$

is the solution of the problem

$$\frac{\partial W_m}{\partial\,\mathrm{Fo}} = \Delta W_m - b^2 W_m + f_m;\ W_m\Big|_{\mathrm{Fo}=0} = 0; \tag{16}$$

$$W_m\Big|_{\Gamma_1} = 0;\qquad \frac{\partial W_m}{\partial \nu}\Big|_{\Gamma_2} = 0; \tag{17}$$

the $\chi_{ij}(x,\ y)$ are coordinate functions which exactly satisfy the boundary conditions (14);

$$\chi_{ij} = \frac{\omega_1}{\omega_1+\omega_2^2}\,\Phi_{ij} = \frac{\omega_1}{\omega_1+\omega_2^2}\Bigg[\left(P_i - \omega_{11}\frac{d\omega_{11}}{dx}\,\frac{dP_i}{dx}\right)\times$$

$$\times\left(V_j - \omega_{12}\frac{d\omega_{12}}{dy}\,\frac{dV_j}{dy}\right) + \left(P_j - \omega_{11}\frac{d\omega_{11}}{dx}\,\frac{dP_j}{dx}\right)\left(V_i - \omega_{12}\frac{d\omega_{12}}{dy}\,\frac{dV_i}{dy}\right)\Bigg]; \tag{18}$$

$$\omega_2 = \omega_{11}\omega_{12};\ \omega_3 = \omega_{21} + \omega_{22} + \sqrt{\omega_{21}^2 + \omega_{22}^2};$$

$$\omega_{11} = x(1-x);\ \omega_{12} = y(1-y);\ \omega_{21} = 0{,}6 - x;\ \omega_{22} = 0{,}6 - y;$$

the $P_i(x)$, $V_i(y)$ are Legendre polynomials.

The coefficients $C_{ijm}(\mathrm{Fo}) \doteqdot \overline{C}_{ijm}(p)$, in accordance with (7), are determined by the formula

$$\overline{C}_{ijm}(p) = \sum_{k,s} Z_{ijks}(p) \int_\Omega [-f_m\chi_{ks}]\,d\Omega;\ Z_{ijks}(p) = \frac{\Delta_{ijks}(p)}{p\Delta(p)}.$$

The inverse transform is realized by the expansion of $Z_{ijks}(p)$ in simple fractions. In finding the roots p_l of the equation $\Delta(p) = 0$, we determine the eigenvalues $p_l^* = -(p_l + b^2)$ of the problem

$$\Delta W - p^* W = 0;\quad W\Big|_{\Gamma_1} = 0;\quad \frac{\partial W}{\partial \nu}\Big|_{\Gamma_2} = 0.$$

The eigenvalues p_l^* ($l = 1, \ldots, n$) are real and positive and form a segment of a monotonically increasing sequence. As the number of coordinate functions increases, the values of p_l^* are stabilized [5].

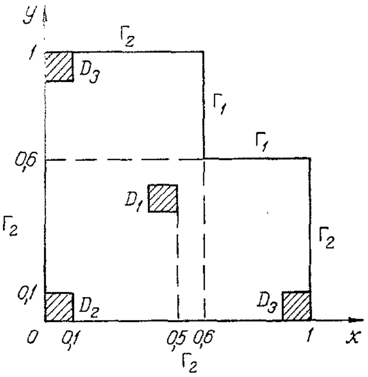

Fig. 1. Geometric scheme of an element with an energy source.

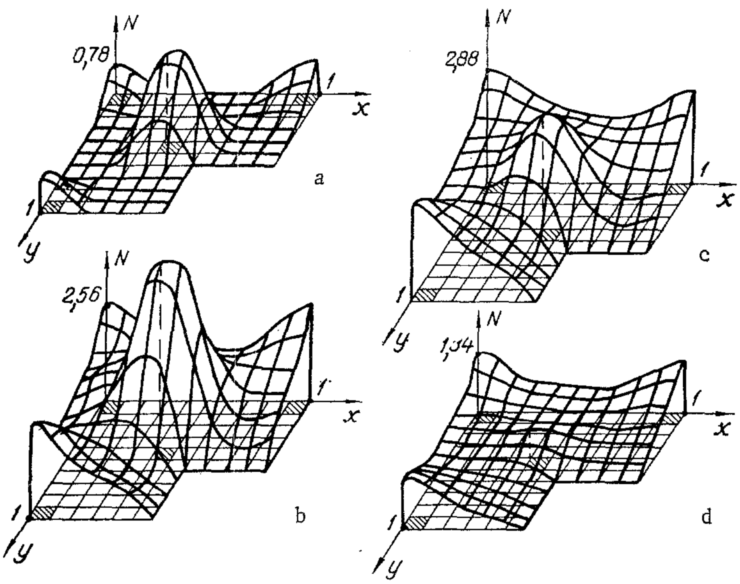

Fig. 2. Distribution of the temperature field in an element (Fig. 1) for: a) Fo = 0.01, b) 0.1, c) 1.0, d) 10.

Figure 2a–d shows in spatial projections the distribution of the temperature field in the form of a dimensionless criterion function $N(x, y, Fo) = \mathring{u}(x, y, Fo)\,\lambda d/P$ of the coordinates x, y, for $Bi = b^2 = 5$; $\varphi_m(Fo) = 1 + A_m \exp(-Fo)$, $A_1 = 19$, $A_2 = 0.5$, $A_3 = 1$; n = 21 when Fo = 0.01, 0.1, 1.0, and 10. For the test of the algorithms we considered the problem with a system of energy sources for a square plate with thermal insulation on its end faces (a plate with no holes cut out); the coordinate functions for this problem were chosen in the form of Φ_{ij} from expression (18). The relative error of the test problem in comparison with the Fourier method for n = 20, $Fo \geqslant 0.01$, did not exceed 2%, and for the test problem and the fundamental problem (13), (14) with small values of Fo, the values of the temperature calculated at the point most remote from the cutout (x = 0, y = 0) practically coincided.

NOTATION

u, temperature; ρ, density; λ, thermal conductivity; c, specific heat capacity; n, number of coordinate functions; $Fo = t(\lambda/\rho c L)$, Fourier number; ν, direction of the inner normal to the contour Γ_2; L, characteristic dimension of the plate; d, thickness of the plate; α, sum of the total heat-transfer coefficients from the surface of the plate; $Bi = \alpha L^2/\lambda d = b^2$, Biot number.

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REMARKS ON WAVE SOLUTIONS OF THE NONLINEAR HEAT-CONDUCTION EQUATION

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Wave solutions of the nonlinear heat-conduction equation are analyzed and their relation to self-similar solutions is established. Solutions of the hyperbolic and the nonlinear heat-conduction equations are compared.

1. Undamped Thermal Waves

Let us consider the nonlinear heat-conduction equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] \tag{1}$$

and compare its wave solutions with solutions of the linear hyperbolic equation

$$\frac{1}{g^2}\frac{\partial^2 T}{\partial t^2} + \frac{c_v\rho}{\lambda_0}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}. \tag{2}$$

According to the data [1], $c_v\rho/\lambda_0 \approx 0.753 \cdot 10^{-3}$ sec/cm^2 for helium at 2°K. In this case, (2) can be replaced by the following

$$\frac{\partial^2 T}{\partial t^2} = g^2\frac{\partial^2 T}{\partial x^2}, \tag{3}$$

which describes the propagation of undamped thermal waves. To find the wave solutions, we go over to the wave variable

$$\xi = v(x) + ct \tag{4}$$

in (1). Let us mention the transformation formula

$$\frac{\partial T}{\partial t} = c\frac{dT}{d\xi};\quad \frac{\partial T}{\partial x} = \frac{dv}{dx}\frac{dT}{d\xi};$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{d^2 v}{dx^2}\frac{dT}{d\xi} + \left(\frac{dv}{dx}\right)^2\frac{d^2 T}{d\xi^2}.$$

Then taking into account that $d^2v/dx^2 = 0$, we will have

$$c - \frac{dk}{dT}\frac{dT}{d\xi}\left(\frac{dv}{dx}\right)^2 = k(T)\frac{d^2 T}{d\xi^2}\left(\frac{dv}{dx}\right)^2 \bigg/ \frac{dT}{d\xi}. \tag{5}$$

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